

3. Single-Degree-of-Freedom (SDOF) Systems: Damped Free Vibration

3.1. Equation of Motion, Critical Damping and Damping Ratio

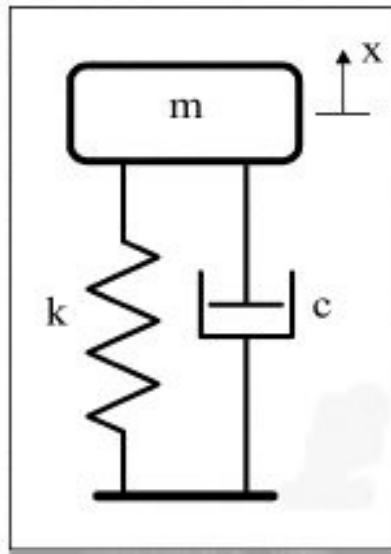


Figure 3.1

Equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (3-1)$$

Critical damping C_c is given by

$$\begin{aligned} C_c &= 2\sqrt{mk} \\ &= 2m\omega \end{aligned} \quad (3-2)$$

Damping ratio (damping factor) ζ is given by

$$\zeta = \frac{c}{C_c} \quad (3-3)$$

[In this notes we use the symbol ζ or ξ to denote damping ratio.]

Systems with $\zeta = 1$, are called critically damped systems.

Systems with $\zeta > 1$, are called over-damped systems.

Systems with $\zeta < 1$, are called under-damped systems.

Critically Damped Systems ($\zeta = 1$)

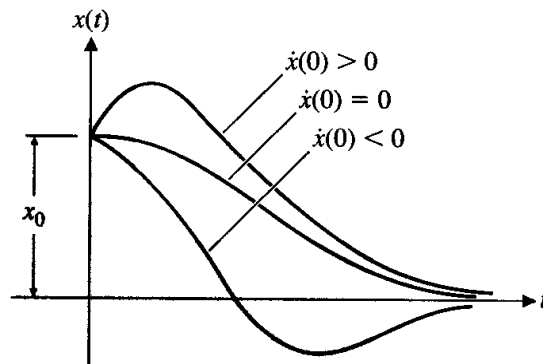


Figure 3.2

Over-damped Systems ($\zeta > 1.0$)

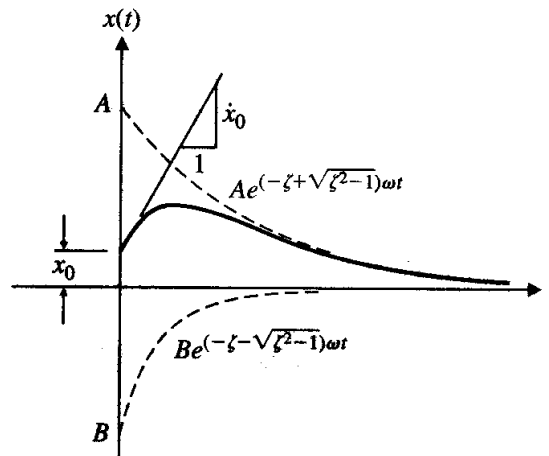


Figure 3.3

Under-damped Systems ($\zeta < 1.0$)

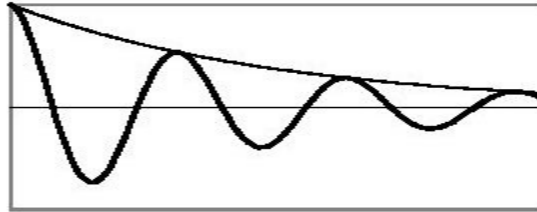


Figure 3.4

3.2. Damped Natural Frequency

Displacement is given by

$$x(t) = X e^{-\zeta \omega t} \sin(\omega_d t + \phi) \quad (3-4)$$

where

X = amplitude of vibration

Φ = phase.

X and Φ are determined by computing $x(t)$ and $v(t)$ at $t=0$ and equating them to initial conditions (initial displacement and initial velocity).

Damped natural frequency is given by

$$\omega_d = \sqrt{1 - \zeta^2} \omega \quad (3-5)$$

3.3. Determination of Damping from Free Vibration Data

Logarithmic decrement method

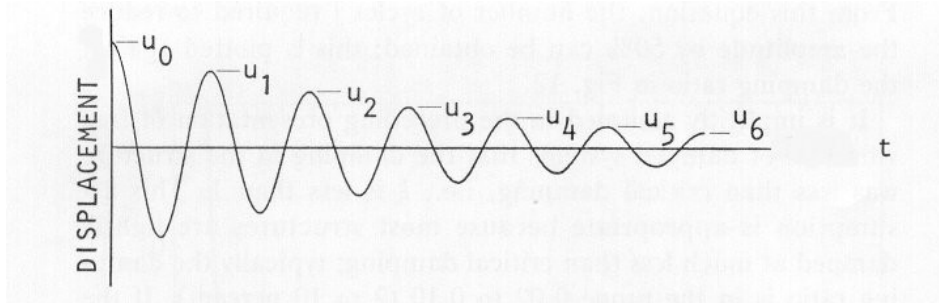


Figure 3.5

Logarithmic decrement (δ) is given by

$$\mathbf{n\delta = \ln (u_i/u_{i+n})} \quad (3-6)$$

Damping ratio (ζ) is given by

$$\mathbf{\delta = \zeta \omega T_D = 2\pi\zeta / (1 - \zeta^2)^{0.5}} \quad (3-7)$$

For small values of ζ , the above equation may be approximated to

$$\mathbf{\delta = 2 \pi \zeta} \quad (3-8)$$

Example 03-01**Analysis of Free Vibration Test Data**Problem:

An equipment that may be treated as a single-degree-of-freedom system was excited by an initial impact, and the free vibration strain was measured. From the decaying sinusoidal time-history of free vibration, the following measurements were made.

- (1) The third positive peak occurs at 0.52 seconds and the eighth positive peak at 0.77 seconds.
- (2) Vibration amplitude at the third cycle is 25 and vibration amplitude at the eighth cycle is 18.

Calculate the period, natural frequency and damping ratio of the system. If the total weight of the equipment is 1000 pounds, what is the system stiffness?

Solution:

Let T_d = damped period.

Time elapsed between the 3-rd and 8-th positive peaks = $5 T_d$
 $= 0.77 - 0.52 = 0.25 \text{ sec}$

So, $T_d = 0.25/5 = 0.05 \text{ sec}$

Damped natural frequency in cycles per second $f_d = 1/T_d$
 $= 1/0.05 = 20 \text{ cps}$

Damped natural frequency in radians per second ω_d

$$= 2\pi f_d$$
$$= 2\pi (20) = 125.6 \text{ rad/sec}$$

Vibration amplitude at the 3-rd cycle $u_3 = 25$

Vibration amplitude at the 8-th cycle $u_8 = 18$

$$n = 8 - 3 = 5$$

Using

$$n\delta = 5\delta = \ln(u_3/u_8) = \ln(25/18) = \ln(1.3889) = 0.3285$$

$$\text{So, } \delta = 0.3285/5 = 0.0657$$

Using Eq. 3.8,

$$\delta = 2\pi\xi = 0.0657$$

$$\text{So, } \xi = 0.0657/2\pi = 0.01046$$